Axiomatic Redistricting*

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Summary. *In a framework with two parties, deterministic voter preferences and geographical constraints, we propose a set of simple axioms and show that they jointly characterize the districting rule that maximizes the number of districts one party can win, given the distribution of individual votes (the "optimal gerrymandering rule"). As a corollary, we obtain that no districting rule can satisfy our axioms and treat parties symmetrically.*

Keywords: districting, gerrymandering.

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1 Introduction

The districting problem has received considerable attention recently, both from the political science and the economics viewpoint. Much of the recent work has focused on strategic aspects and the incentives induced by different institutional designs on the political parties, legislators and voters (see, among others, Besley and Preston, 2007, Friedman and Holden, 2008, Gul and Pesendofer, 2007). Other contributions have looked at the welfare implications of different redistricting policies (e.g. Coate and Knight, 2007). Finally, there is also a sizable literature on the computational aspects of the districting problem (see, e.g. Puppe and Tasnádi, 2008, and the references therein).

In contrast to these contributions, the present paper takes a *normative* point of view. We formulate desirable properties ("axioms") and investigate which districting functions satisfy them. There are several reasons for exploring this approach. First, the axiomatic method allows one to endow the vast space of conceivable districting rules with useful additional structure: each combination of desirable properties characterizes a specific class of districing rules, and thereby helps one to assess their respective merits. Second, one may hope that specific combinations of axioms single out a few, perhaps sometimes even a unique districting rule, thus reducing the space of possibilities. Finally, the axiomatic approach may reveal incompatibility of certain axioms by showing that *no* districting rule can satisfy certain combinations of desirable properties, thereby terminating a futile search.

In a framework with two parties and geographical constraints on the shape of districts, we propose a set of simple axioms and show that they jointly characterize the districting rule that maximizes the number of districts one party can win, given the distribution of individual votes (the "optimal gerrymandering rule"). While some of the axioms have a more pragmatic justification, others have straightforward normative foundations such as the neutrality property which requires that a districting rule should treat parties symmetrically. Evidently, by generating a maximal number of winning districts for one of the parties, the optimal gerrymandering rule violates the neutrality axiom. Therefore, as a straightforward corollary of our main result, we obtain that no districting rule can satisfy a set of reasonable properties and treat parties symmetrically at the same time.

The work closest to ours in the literature is Chambers (2008, 2009) who also takes an axiomatic approach. However, one of his central conditions is the requirement that the election outcome be *independent* of the way districts are formed ("gerrymandering-proofness"), and the main purpose of his analysis is to explore the consequences of this requirement. By contrast, our focus is precisely on the districting process which we try to structure by means of simple governing principles. In particular, geographical constraints which are absent in Chambers' model play an important role in our analysis.

The districting rules that we consider depend among other things on the distribution of votes for each party in the population. One might argue, perhaps on grounds of some "absolute" notion of *ex ante* fairness, that a districting rule must not depend on voters' party preferences since these can change over time. From this perspective, the districting problem is not really an issue and it would seem that any districting which partitions the population in (roughly) equally sized subgroups should be acceptable. By contrast, in the present paper we are interested in a "relative" or *ex post* notion of fair districting, i.e. in the question of what would constitute an acceptable districting rule *given* the distribution of the supporters of each party in the population. This question seems particularly important for practical purposes since a redistricting policy can be successfully implemented only if it receives sufficient support by the *actual* legislative body.

2 The Framework

We assume that parties A and B compete in an electoral system consisting only of single member districts, where the representatives of each districts are determined by plurality. The parties as well as the independent bodies face the following districting problem.

Definition 1 (Districting problem)**.** A *districting problem* is given by the structure $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$, where

- the voters are located within a subset X of the plane $\mathbb{R}^2$,
- $\mathcal{A}$ is the σ-algebra on X consisting of all districts that can be formed without geographical or any other type of constraints,
- the distribution of voters is given by a measure μ on $(X, \mathcal{A})$,
- the distributions of party A and party B voters are given by measures μ_A and μ_B on $(X, \mathcal{A})$ such that $\mu = \mu_A + \mu_B$,
- t is the given number of seats in parliament,
- $G \subseteq \mathcal{A}$, also called *geography*, is the set of admissible districts satisfying $\mu(g) = \mu(X)/t$ and
$$\mu_A(g) \neq \mu_B(g) \tag{1}$$
for all $g \in G$, and possessing a partitioning of X, i.e there exist mutually disjoint sets $g'_1, \ldots, g'_t \in G$ such that $\cup_{i=1}^t g'_i = X$.

(1) excludes the possibility of a draw in any district in order to simplify our analysis by avoiding the introduction of tie-breaking rules. (1) is satisfied, for instance, if we have a finite set of voters, μ, μ_A, μ_B are counting measures and the district sizes are odd.

Definition 2 (Districting)**.** A *districting* for the problem $(X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is a subset $D \subseteq G$ such that D forms a partition of X and $\#D = t$.

We shall denote by $\delta_A(D)$ and $\delta_B(D)$ the number of districts won by party A and party B under D, respectively. We write $\mathcal{D}_\Pi$ for the set of all districtings of problem Π and let $\delta_A(\mathcal{D}) = \{\delta_A(D) : D \in \mathcal{D}\}$ and $\delta_B(\mathcal{D}) = \{\delta_B(D) : D \in \mathcal{D}\}$ for any $\mathcal{D} \subseteq \mathcal{D}_\Pi$.

Definition 3 (Solution)**.** A *solution* F associates to each districting problem Π a non-empty set of admissible districtings $F_\Pi \subseteq \mathcal{D}_\Pi$.

The case of two districts plays a fundamental role in our analysis. Note that by (1) it is not possible that a party can win both districts under one districting and lose both districts under another districting, i.e. if $t = 2$ then $\delta_A(\mathcal{D}_\Pi)$ (respectively, $\delta_B(\mathcal{D}_\Pi)$) cannot contain both 0 and 2. Our first axiom requires that a solution must in fact be "determinate" in the two-district case, i.e. if one party wins both districts under one admissible districting it must win both districts under *all* admissible districtings. The underlying intuition is that, at least in the two district case, a solution should not leave open the issue whether or not there is a draw between the two parties in the parliament.

Axiom 1 (Two District Determinacy)**.** A solution F satisfies *two district determinacy* if for any districting problem Π with $t = 2$ we have that $\delta_A(D) = 2$ for some $D \in F_\Pi$ implies $\delta_A(F_\Pi) = \{2\}$, and $\delta_B(D) = 2$ for some $D \in F_\Pi$ implies $\delta_B(F_\Pi) = \{2\}$.

The second axiom states that if a possible districting induces the same number of winning districts for each party than some admissible districting, it must be admissible as well.

Axiom 2 (Indifference)**.** A solution F satisfies *indifference* if for any districting problem Π we have that $D \in F_\Pi$, $D' \in \mathcal{D}_\Pi$, $\delta_A(D) = \delta_A(D')$ and $\delta_B(D) = \delta_B(D')$ implies $D' \in F_\Pi$.

The third axiom represents a simple extension property when the geography is expanded in such a way that the number of winning districts for the parties does not change.

Axiom 3 (Extendibility)**.** If for any districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ and any partitioning $D = \{g_1, \dots, g_t\} \notin \mathcal{D}_\Pi$ of X for which $D \subseteq \mathcal{A}$, g_i satisfies (1) for any $i = 1, \dots, t$, $\mathcal{D}_\Pi \cup \{D\} = \mathcal{D}_{\Pi'}$, where $\Pi' = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G \cup D)$, and $\delta_A(D) \in \delta_A(F_\Pi)$ we have that $F_\Pi \cup \{D\} = F_{\Pi'}$, then we call solution F *extendible*.

Our next axiom requires that a solution to a problem should also deliver solutions to specific subproblems. Its spirit is very similar to the *uniformity principle* in Balinski and Young's (2001) theory of apportionment ("every part of a fair division should be fair").

Axiom 4 (Consistency)**.** A solution F satisfies *consistency* if for any districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$, any $D \in F_\Pi$ and any $D' \subseteq D$ we have for $Y = \cup_{d \in D'} d$ that

$$D|_Y = D' \in F_{\Pi|_Y} = F_{(Y, \mathcal{A}|_Y, \mu|_Y, \mu_A|_Y, \mu_B|_Y, \#D', G|_Y)},$$

where $\mathcal{A}|_Y = \{A \cap Y : A \in \mathcal{A}\}$, $G|_Y = \{g \in G : g \subseteq Y\}$ and $\mu|_Y, \mu_A|_Y, \mu_B|_Y$ stand for the restrictions of measures μ, μ_A, μ_B to $(Y, \mathcal{A}|_Y)$.

We also require that a solution selects districtings based on the number of winning districts in a transitive way.[1]

Axiom 5 (Transitivity)**.** If for any two districting problems $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ and $\Pi' = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G')$ for which $\mathcal{D}_\Pi \cup \mathcal{D}_{\Pi'} = \mathcal{D}_{\Pi^*}$, where $\Pi^* = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G \cup G')$, we have that $D_1 \in F_\Pi$, $D_2 \in \mathcal{D}_\Pi$, $D_2 \notin F_\Pi$, $D_2 \in F_{\Pi'}$, $D_3 \in \mathcal{D}_{\Pi'}$ and $D_3 \notin F_{\Pi'}$ implies

$$D_1 \in F_{\Pi^*} \text{ and } D_3 \notin F_{\Pi^*},$$

then solution F is *transitive*.

[1] It is worthwhile mentioning that our main characterization result below uses the axioms of extendibility, consistency and transitivity only in cases in which there are exactly two districts. We could thus restrict these axioms in the same way as the first axiom and require them to hold only for $t = 2$ in case of extendibility and transitivity, and for $\#D' = 2$ in case of consistency, respectively.

The final axiom expresses a fundamental principle of fairness in our context, namely the symmetric treatment of parties *ex ante*.

Axiom 6 (Neutrality)**.** A solution F satisfies *neutrality* if for any districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ and any $D \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,t,G)}$ it follows that $D \in F_{(X,\mathcal{A},\mu,\mu_B,\mu_A,t,G)}$.

In the following we will show that for a large class of geographies no solution can satisfy all six axioms simultaneously. While we consider the neutrality condition to be an indispensable fairness requirement, our proof strategy is to show that the first five axioms characterize the "optimal partisan gerrymandering" solution defined in the next section. Since this solution evidently violates the neutrality requirement the impossibility result follows.

3 Two solutions

The following solution determines the optimal partisan gerrymandering from the viewpoint of party A.

Definition 4 (Optimal solution for A)**.** The optimal solution O^A for party A determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of those districtings that maximize the number of winning districts for party A, i.e.

$$O^A_\Pi = \arg \max_{D \in \mathcal{D}_\Pi} \delta_A(D).$$

The optimal solution O^B for party B is defined analogously. If we are referring to an optimal solution O, then we have either O^A or O^B in mind.

The optimal solution satisfies two district determinacy, indifference, extendibility and transitivity by definition. Moreover, O^A satisfies consistency; indeed, otherwise there would exist $D' \subset D \in O^A_\Pi$ such that $D' \notin O^A_{\Pi|_Y}$, where $Y = \cup_{d \in D'} d$, which in turn would imply that $\delta_A(D'' \cup (D \setminus D')) > \delta_A(D)$ for any $D'' \in O^A_{\Pi|_Y}$, a contradiction. On the other hand, the optimal solution evidently violates neutrality since an optimal solution for one party in general differs from any optimal solution for the other party.

The following alternative solution treats both parties equally.

Definition 5 (Most equal)**.** The solution ME determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of most equal districtings, i.e.

$$ME_\Pi = \arg \min_{D \in \mathcal{D}_\Pi} |\delta_A(D) - \delta_B(D)|. \tag{2}$$

Since an equal solution does not always exist the most equal solution aims to get as close as possible to equality in terms of winning districts for the two parties.

ME satisfies two district determinacy, indifference, extendibility, transitivity and neutrality by definition. However, ME violates consistency. Consider, for instance, a 10 to 10 districting and let Y be the union of the 10 winning districts for party A to define a subproblem. In general, there will exist a districting problem with $t = 20$ such that the subproblem has a more equal solution than the 10 to 0 for party A. Thus, ME does not satisfy consistency.

4 A Characterization Result and an Impossibility

First, we consider districting problems with only two districts and we show that extendibility, indifference and transitivity implies the following property.

Definition 6. A solution F satisfies *two district consistency* if for any problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, 2, G)$ and any party $C \in \{A, B\}$

(i) the existence of two districtings $D, D' \in \mathcal{D}_\Pi$ satisfying $\delta_C(D) = 1$, $\delta_C(D') = 2$ and $D' \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,D\cup D')}$ implies for any two districtings $E, E' \in \mathcal{D}_\Pi$ satisfying $\delta_C(E) = 1$ and $\delta_C(E') = 2$ that $E' \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,E\cup E')}$;

(ii) the existence of two districtings $D, D' \in \mathcal{D}_\Pi$ satisfying $\delta_C(D) = 1$, $\delta_C(D') = 2$ and $D \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,D\cup D')}$ implies for any two districtings $E, E' \in \mathcal{D}_\Pi$ satisfying $\delta_C(E) = 1$ and $\delta_C(E') = 2$ that $E \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,E\cup E')}$.

Lemma 1. *Extendibility, indifference and transitivity implies two district consistency.*

Proof. We only show that extendibility, indifference and transitivity implies point (i) of Definition 6 since point (ii) can be established in an analogous way. Suppose that there exist districtings D, D', E, E' such that $\delta_C(D) = 1$, $\delta_C(D') = 2$, $\delta_C(E) = 1$, $\delta_C(E') = 2$, $D' \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,D\cup D')}$ and $E' \notin F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,E\cup E')}$. Since $E \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,E\cup E')}$ it follows by extendibility that

$$\{D, E\} = F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,D\cup E\cup E')}$$

in case of $D \neq E$. Now employing transitivity for $D_1 = D'$, $D_2 = D$, $D_3 = E'$, $G = \{D, D'\}$ and $G' = \{D, E, E'\}$, we obtain that $D' \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,G\cup G')}$ and $E' \notin F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,2,G\cup G')}$, which in turn is in contradiction with indifference.

In case of $D = E$ we do not need extendibility and can apply transitivity immediately by setting $D_1 = D'$, $D_2 = D$, $D_3 = E'$, $G = \{D, D'\}$ and $G' = \{D, E'\} = \{E, E'\}$. □

Lemma 2. *The optimal and the most equal solutions satisfy two district consistency.*

Proof. Since O^A maximizes the number of winning districts for party A, it will always choose the districting with the larger number of winning districts. The same holds for O^B. Thus, O satisfies two district consistency. Finally, since ME selects always a one to one solution whenever possible it satisfies two district consistency. □

Lemma 3. *If F satisfies two district determinacy and two district consistency, then $F = O$ or $F = ME$ for $t = 2$.*

Proof. Since there cannot be two districtings D and D' with $\delta_A(D) = 2$ and $\delta_B(D') = 2$, ME satisfies two district determinacy. For the case of $C = A$ in the statement of two district consistency we obtain that either $F = O^A$ or $F = ME$ for $t = 2$ since ME chooses a one to one districting whenever possible. Similarly, for $C = B$ we obtain that either $F = O^B$ or $F = ME$. □

Consider districting problems for $t = 3$ with the 9 possible districts and the 3 resulting districtings shown in Figure 1, in which party A voters are indicated by empty circles and party B voters by solid circles, μ equals the counting measure on

$(X, \mathcal{A})$ and μ_A, μ_B determine the respective number of party A and party B voters. It can be verified that considering the districtings from left to right we obtain 3 to 0, 2 to 1 and 1 to 2 winning districtings for party A, respectively. Thus, the optimal solution for party A would choose the first districting from the left. However, by assuming that F equals O^A for $t = 2$ and that F is consistent we cannot rule out a solution that selects, for instance, the 1 to 2 districting. Hence, consistency cannot restrict the set of solutions on arbitrary geographies. Therefore, we have to restrict the set of possible geographies in a reasonable way.

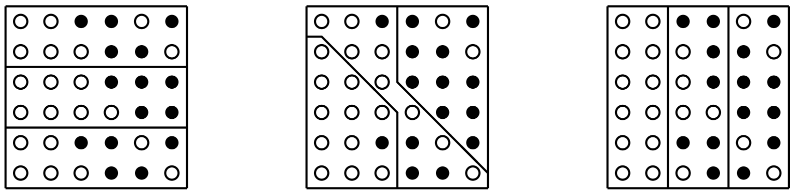

Figure 1: Unlinked districtings

Definition 7. The geography G of a problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is *linked* if for any two possible districtings $D, D' \in \mathcal{D}_\Pi$ there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D = D_1$, $D' = D_k$ and $\#D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k-1$.

We provide a natural example of a linked geography in the appendix to underline that linkedness is an acceptable assumption that can be imposed on geographies.

Proposition 1. *If F equals O^A for $t = 2$ and F is consistent and indifferent, then $F = O^A$ for linked geographies.*

Proof. Take a districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ with $t \geq 3$ and suppose that $F_\Pi \neq O^A_\Pi$ but F is consistent and indifferent. Since F_Π is not O^A_Π, there exist $D' \in O^A_\Pi$ and $D \in F_\Pi$ such that $\delta_A(D') > \delta_A(D)$ by indifference. Since Π has a linked geography there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D' = D_1$, $D = D_k$ and $\#D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k-1$. Let $i' = \max\{i \in \{1, \ldots, k-1\} : \delta_A(D_1) = \delta_A(D_2) = \ldots = \delta_A(D_i) > \delta_A(D_{i+1})\}$ and $j' = \min\{j \in \{2, \ldots, k\} : \delta_A(D_{j-1}) \neq \delta_A(D_j) = \ldots = \delta_A(D_k)\}$. It follows by indifference that $D_{i'} \in O^A_\Pi$ and $D_{j'} \in F_\Pi$.

If $i' = j' - 1$, then $D_{i'}$ and $D_{j'}$ just differ in two districts, which we shall denote by d, d', e and e', where the first two districts belong to $D_{i'}$ while the latter two to $D_{j'}$. Observe that $D_{i'} \setminus \{d, d'\} = D_{j'} \setminus \{e, e'\}$ by linkedness. Let $Y = d \cup d' = e \cup e'$. Since O^A and F are consistent we have $\{d, d'\} \in O^A_{\Pi|_Y}$ and $\{e, e'\} \in F_{\Pi|_Y}$. Our assumption that F equals O^A for $t = 2$ and $D_{i'} \setminus \{d, d'\} = D_{j'} \setminus \{e, e'\}$ implies $\delta_A(D_{i'}) = \delta_A(D_{j'})$; a contradiction.

Assume that $i' < j' - 1$. Employing (1), consistency and linkedness, we have

$$|\delta_A(D_i) - \delta_A(D_{i+1})| \leq 1 \tag{3}$$

for all $i = i', \ldots, j' - 1$ because D_i and D_{i+1} just differ in two districts. Moreover, by the definition of j', by consistency and by our assumption that F equals O^A for $t = 2$ we must have $\delta_A(D_{j'-1}) < \delta_A(D_{j'})$, which in turn implies by (3) and $\delta_A(D_{i'}) > \delta_A(D_{j'-1})$

that there exists a smallest $j^* \in \{i'+1, \ldots, j'\}$ such that $\delta_A(D_{j^*}) = \delta_A(D_{j'})$. Clearly, $D_{j^*} \in F_\Pi$ by indifference. We cannot have $j^* > i'+1$ since this would contradict the definition of j^*, $D_{j^*} \in F_\Pi$, $\delta_A(D_{j^*-1}) < \delta_A(D_{j^*})$ and (3). However, if $i' = j^* - 1$, then we can repeat the argument of the previous paragraph by replacing j' with j^* to obtain a contradiction. □

Since the most equal solution does not satisfy consistency we cannot extend ME for $t = 2$ to arbitrary t in a manner of Proposition 1. However, it might be the case that ME for $t = 2$ can be extended to another consistent solution. The next proposition demonstrates that such an extension does not exist.

Proposition 2. *There does not exist a consistent and indifferent solution F that equals ME for $t = 2$ even for linked geographies.*

Proof. Suppose that there exists a consistent and indifferent solution F that equals ME for $t = 2$. Pick a districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, 3, G)$, where X consists of 27 voters, $\mathcal{A}$ equals the set of all subsets of X, $G = \{d_1, \ldots, d_9\}$ is shown in Figure 2 in which party A voters are indicated by empty circles and party B voters by solid circles, μ equals the counting measure on $(X, \mathcal{A})$ and μ_A, μ_B determine the respective number of party A and party B voters.

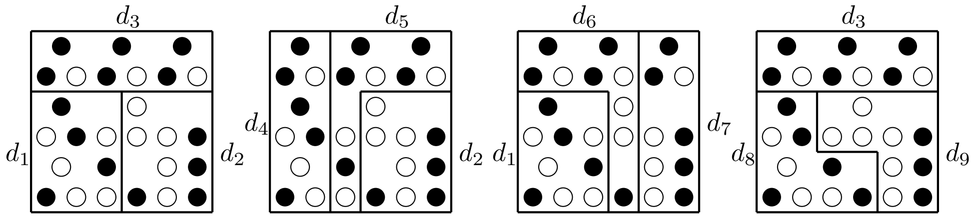

Figure 2: ME cannot be extended

We can see from Figure 2 that the four possible districtings are $D_1 = \{d_1, d_2, d_3\}$, $D_2 = \{d_2, d_4, d_5\}$, $D_3 = \{d_1, d_6, d_7\}$ and $D_4 = \{d_3, d_8, d_9\}$. It can be checked that the given geography is linked. Since $\delta_A(D_1) = 2$ and $\delta_A(D_2) = \delta_A(D_3) = \delta_A(D_4) = 1$ we must have either $\{D_1\} = F_\Pi$, $\{D_2, D_3, D_4\} = F_\Pi$ or $\{D_1, D_2, D_3, D_4\} = F_\Pi$ by indifference. First, consider the cases of $\{D_1\} = F_\Pi$ and $\{D_1, D_2, D_3, D_4\} = F_\Pi$. By consistency we must have $\{d_1, d_2\} \in F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$, where $X' = d_1 \cup d_2$, $G' = \{d_1, d_2, d_8, d_9\}$ and $\mathcal{A}'$, μ', μ'_A, μ'_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X', respectively. However, $F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$ should equal $\{d_8, d_9\}$ since $F = ME$ for $t = 2$; a contradiction. Second, consider the case of $\{D_2, D_3, D_4\} = F_\Pi$ and pick the case of D_3. By consistency we must have $\{d_6, d_7\} \in F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$, where $X'' = d_6 \cup d_7$, $G'' = \{d_2, d_3, d_6, d_7\}$ and $\mathcal{A}''$, μ'', μ''_A, μ''_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X'', respectively. However, $F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$ should equal $\{d_2, d_3\}$ since $F = ME$ for $t = 2$; a contradiction. □

Our main theorem follows from Lemmas 1, 2, 3 and Propositions 1 and 2.

Theorem 1. *The optimal solution O is the only solution that satisfies two district determinacy, indifference, extendibility, consistency and transitivity on linked geographies.*

We obtain the following result as a simple corollary.

Corollary 1. *There does not exist a two district determinate, indifferent, extendible, consistent, transitive and neutral solution on linked geographies.*

Appendix

We provide an example showing that linkedness is satisfied by a quite natural planar geography. A bounded subset A of $\mathbb{R}^2$ will be called *strictly connected* if its boundary ∂A is a Jordan curve. A subset A of a strictly connected set $B \subseteq \mathbb{R}^2$ *separates* B if $B \setminus A$ is not strictly connected. We call a continuous function $f : X \to \mathbb{R}$ *nowhere constant* if for any $x \in X$ and any neighborhood $N(x)$ of x there exists a $y \in N(x)$ such that $f(x) \neq f(y)$.

Example 1 (Regular). A districting problem $\Pi = (X, \mathcal{B}(X), \mu, \mu_A, \mu_B, t, G)$ is called *regular* if

1. X is a bounded and strictly connected subset of $\mathbb{R}^2$,
2. μ, μ_A and μ_B are finite and absolutely continuous measures on $(X, \mathcal{B}(X))$ with respect to the Lebesgue measure,
3. G consists of all bounded, strictly connected and $\mu(X)/t$ sized subsets of $\mathcal{B}(X)$ and
4. there exists a continuous nowhere constant function $f : X \to \mathbb{R}$ such that $\mu_A(C) = \int_C f(\omega) d\mu(\omega)$ for all $C \in \mathcal{B}(X)$.

The last assumption is a purely technical one providing a sufficient condition to ensure that the districtings emerging in the proof of Lemma 4 can be selected in a way that they satisfy (1).

In what follows we write $D \sim D'$ if for two districtings $D, D' \in \mathcal{D}_\Pi$ there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D = D_1$, $D' = D_k$ and $\# D_i \cap D_{i+1} = t-2$ for all $i = 1, \ldots, k-1$.

Lemma 4. *Regular districting problems are linked.*

Proof. Linkedness is clearly satisfied if $t = 1$ or $t = 2$. We show that the linkedness of all regular districting problems for $t \leq n$ implies the linkedness of all regular districting problems for $t = n + 1$, which yields by induction the proof of our statement.

Take two arbitrary districtings D and E of a districting problem with $t = n + 1$. We can pick a district $d \in D$ such that d and X have at least a non-degenerate curve as a common boundary and d does not separate X, i.e. there exists a curve C of positive length such that $C \subseteq \partial d \cap \partial X$ and $X \setminus d$ remains strictly connected. Moreover, there exist a district $e \in E$ and a curve $C \subset \mathbb{R}^2$ of positive length such that $\mu(d \cap e) > 0$ and $C \subseteq \partial d \cap \partial e \cap \partial X$.

Case 1: Assume that e does not separate X. Since μ is absolutely continuous there exists a set h such that $\mu(h) = 2\mu(X)/(n + 1)$, $d \cup e \subset h$, $d' = h \setminus d \in G$ and $e' = h \setminus e \in G$ and h does not separate X. Let H be a districting of $Y = X \setminus h$ into $n - 1$ strictly connected districts. Then $\Pi\ |_{Y \cup d'}$ and $\Pi\ |_{Y \cup e'}$ are regular districting problems, and therefore it follows by the induction hypothesis that $D \sim H \cup \{d, d'\}$ and $H \cup \{e, e'\} \sim E$. Clearly $\{d, d'\} \sim \{e, e'\}$, which gives $H \cup \{d, d'\} \sim H \cup \{e, e'\}$.

Case 2: Assume that e does separate X, where the number of strictly disconnected regions of $X \setminus \{e\}$ equals $k \leq n$. Then $d^c \cap \partial e \cap \partial X \neq \emptyset$. We can find a district $e' \in E$ with a unique boundary element $x \in \partial e'$ satisfying $x \in d^c \cap \partial e \cap \partial X$ and that $\partial e \cap \partial e'$ has a common curve of positive length starting from x. Hence, one can exchange territories between e and e' so that for the resulting new districts h and h' we have that $d \cap e \subset h$, h separates X into at most $k-1$ strictly disconnected regions. Clearly, $E' = (E \setminus \{e, e'\}) \cup \{h, h'\} \sim E$ and we can continue with either Case 1 or Case 2, where now E' and h plays the role of E and e, respectively. Observe that we arrive to Case 1 after at most k steps. □

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