Egy sztochasztikus dominancián alapuló döntési modell

Fábián Csaba
Kecskeméti Föiskola
és ELTE

Gautam Mitra
CARISMA,
Brunel University

Diana Roman
CARISMA,
Brunel University

Victor Zviarovich
CARISMA,
Brunel University

Overview

Introduction: Second-order Stochastic Dominance.
- SSD-based portfolio-optimisation models.
- Focus on the multi-objective SSD model of Roman, Darby-Dowman, and Mitra (2006).

Solution methods for the multi-objective SSD model
- Direct solution of large LP problem, cutting-plane approach, and regularisation.
- Computational study: comparison of solution times.

Enhanced version of the multi-objective SSD model: Scaling objectives.
- Computational study: comparison of optimal portfolio return distributions.
- Minimisation of a convex risk measure; connection with robust optimisation.

Conclusion and future prospects.

Introduction: *Second-order Stochastic Dominance (SSD)*

R, R' : random numbers (returns of different investments).

$R \succeq_{sso} R'$ iff $\mathrm{E}(U(R)) \geq \mathrm{E}(U(R'))$ holds
with any *monotonic and concave* utility function U.

$R \succ_{sso} R'$ iff $R \succeq_{sso} R'$ and $R' \not\succeq_{sso} R$.

Introduction: *Notation for SSD-based portfolio-optimisation models*

n : number of assets into which we may invest.

$\boldsymbol{R} = (R_1, \ldots, R_n)^T$: returns of the different assets (random vector).

$\boldsymbol{x} = (x_1, \ldots, x_n)^T$: portfolio; capital invested in different assets (decision variable).
Feasible portfolios: $\boldsymbol{x} \in X$.

$R_{\boldsymbol{x}} = \boldsymbol{R}^T \boldsymbol{x}$: yield of portfolio $\boldsymbol{x}$.
(Random number in case decision is made on portfolio.)

$\boldsymbol{x}^* \in X$ efficient portfolio iff there is no $\boldsymbol{x} \in X$ such that $R_{\boldsymbol{x}} \succ_{SSD} R_{\boldsymbol{x}^*}$.

$\widehat{R}$: benchmark yield (random number). Possibly the yield of a benchmark portfolio.

Introduction: ***SSD-constrained model*** Dentcheva and Ruszczyński (2006)

$$\max \ \mathrm{E}(R_{\boldsymbol{x}})$$

$$\text{such that} \quad \boldsymbol{x} \in X,$$

$$R_{\boldsymbol{x}} \succeq_{SSD} \widehat{R}.$$

Introduction: *Multi-objective SSD model* Roman, Darby-Dowman, and Mitra (2006)

Assume discrete finite distributions with equally probable outcomes.

Equivalent definition of Second-Order Stochastic Dominace:

R, R' : random numbers (returns of different investments);

$$r_1 \le \ldots \le r_S \quad : \quad \text{ordered outcomes of } R,$$

$$r'_1 \le \ldots \le r'_S \quad : \quad \text{ordered outcomes of } R'.$$

$$R \succeq_{ssd} R' \quad \text{is equivalent to} \quad \underbrace{r_1 + \ldots + r_i}_{\text{tail}_i(R)} \ge \underbrace{r'_1 + \ldots + r'_i}_{\text{tail}_i(R')} \quad \text{for} \quad i = 1, \ldots, S.$$

Remark:

$\frac{1}{S}\,\text{tail}_i(R)$ is the *unconditional* expectation of the least $\frac{i}{S} * 100\%$ outcomes of R.

Introduction: ***Multi-objective SSD model*** Roman, Darby-Dowman, and Mitra (2006)

Model formulation: $$\max_{\boldsymbol{x}\in X} \Big(\text{tail}_1(R\boldsymbol{x}),\ \ldots,\ \text{tail}_S(R\boldsymbol{x}) \Big),$$

multi-objective maximization is considered with respect to

the reference point $$\widehat{\tau} := \Big(\text{tail}_1(\widehat{R}),\ \ldots,\ \text{tail}_S(\widehat{R}) \Big).$$

Putting into practice: define and maximise achievement function:

$$\max_{\boldsymbol{x}\in X} \Gamma(\boldsymbol{x}), \qquad \text{where} \quad \Gamma(\boldsymbol{x}) := \min_{1\le i\le S} \Big(\text{tail}_i(R\boldsymbol{x}) - \widehat{\tau}_i \Big).$$

Introduction: ***Multi-objective SSD model*** Roman, Darby-Dowman, and Mitra (2006)

Computational experience: optimal portfolio robustly outperforms stock index.

Solution methods for the multi-objective SSD model: *Direct approach*

$R_{\boldsymbol{x}} = R^T \boldsymbol{x}$. Realisations of the random vector $\boldsymbol{R}$: $\boldsymbol{r}^{(1)}, \ldots, \boldsymbol{r}^{(S)}$
(each occurring with prob. $1/S$).

Computation of tails, using Rockafellar and Uryasev (2000) :

$$\mathrm{tail}_i\,(R_{\boldsymbol{x}}) \;=\; \max_{t \in \mathbb{R}} \; i\,t - \sum_{j=1}^{S} \left[\, t - \boldsymbol{r}^{(j)\,T} \boldsymbol{x} \,\right]_+ .$$

Lifting representation as linear programming problem:

$$\mathrm{tail}_i\,(R_{\boldsymbol{x}}) \;=\; \max \; i\,t - \sum_{j=1}^{S} d_j$$

such that $t,\; d_1, \ldots, d_S \in \mathbb{R}$,

$$d_j \geq t - \boldsymbol{r}^{(j)\,T} \boldsymbol{x}, \quad d_j \geq 0 \qquad (j = 1, \ldots, S).$$

Solution methods for the multi-objective SSD model: *Direct approach*

Implementation:

- Lifting representation : Multi-objective model represented as LP having S^2 new variables.

- Resulting LP problem solved by CPLEX.

Experience: only small problems could be solved in realistic time (number of assets: $n = 76$, number of scenarios: $S \leq 600$).

Solution methods for the multi-objective SSD model: *Cutting plane approach*

Computation of tails, using Rockafellar and Uryasev (2000) :

$$\text{tail}_i(R_{\boldsymbol{x}}) = \max_{t \in \mathbb{R}} \; it - \sum_{j=1}^{S} \left[t - \boldsymbol{r}^{(j)\,T}\boldsymbol{x} \right]_+ .$$

Cutting-plane representation using Künzi-Bay and Mayer (2006) :

$$\text{tail}_i(R_{\boldsymbol{x}}) = \min \sum_{j \in \mathcal{J}} \boldsymbol{r}^{(j)\,T}\boldsymbol{x}$$

$$\text{such that} \quad \mathcal{J} \subset \{1, \ldots, S\}, \; |\mathcal{J}| = i.$$

Astronomical number of cuts!
But in a cutting-plane method, we only need a few of them.

Solution methods for the multi-objective SSD model: *Cutting plane approach*

Cutting-plane representation of the multi-objective problem:

$$\begin{array}{ll} \max & \vartheta \\ \text{such that} & \vartheta \in \mathbb{R}, \quad \boldsymbol{x} \in X, \\ & \vartheta \ \leq \ \sum_{j \in \mathcal{J}_i} \boldsymbol{r}^{(j)\,T}\boldsymbol{x} \ - \widehat{\tau}_i \qquad \text{for each} \quad \mathcal{J}_i \subset \{1, \ldots, S\}, \ \ |\mathcal{J}_i| = i, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{where} \quad i = 1, \ldots, S. \end{array}$$

Cutting-plane method can be effectively implemented: given $\overline{\boldsymbol{x}}$,
deepest cut can be constructed by simply sorting the outcomes $\boldsymbol{r}^{(1)\,T}\overline{\boldsymbol{x}}, \ \ldots, \ \boldsymbol{r}^{(S)\,T}\overline{\boldsymbol{x}}$.

Solution methods for the multi-objective SSD model: *Cutting plane approach*

Structure of the special cutting-plane solver developed:

- Cut generation implemented in C.

- Cutting-plane model problems formulated by
  AMPL modelling system (Fourer, Gay and Kernighan 1989),
  using AMPL COM Component Library (Sadki 2005).

- Solver: FortMP (Ellison, Hajian, Levkovitz, Maros, Mitra 1999).

Solution methods for the multi-objective SSD model: *Cutting plane approach*

We solved a score of problems with $n = 76$ assets and up to $S = 30,000$ scenarios.

For $S \leq 10,000$ usable near-optimal solutions were found in 30 secs
(using loose relative stopping tolerance).

With absolute stopping tolerance set to $\epsilon = 1e - 7$,
solution time was less than 16 mins.

Solution methods for the multi-objective SSD model: *Cutting plane approach*

Typical iteration counts (for $n = 76$ assets, stopping tolerance set to $\epsilon = 1e-7$) :

scenarios	cutting plane iterations
5,000	71
7,000	83
10,000	73
15,000	91
20,000	100
30,000	96

Solution methods for the multi-objective SSD model: *Regularised cutting plane*

Bundle-type method applied for the solution of the master problem:
Level Method of Lemaréchal, Nemirovskii, Nesterov (1995).

Favourable experience with variants of Level Method adapted to stochastic programming:

- Fábián and Szőke (2007),
- Zviarovich, Ellison, Fábián, Mitra.

Solution methods for the multi-objective SSD model: *Regularised cutting plane*

We also solved our testproblems with the regularised method.

($n = 76$ assets, stopping tolerance set to $\epsilon = 1e-7$, level parameter set to 0.5.)

Typical iteration counts:

scenarios	pure cutting plane iterations	regularised iterations
5,000	71	23
7,000	83	27
10,000	73	28
15,000	91	24
20,000	100	27
30,000	96	27

Experimental estimate on iteration count: $O\left(n \ln \frac{1}{\epsilon} \right)$.

Enhanced multi-objective SSD model: *Motivation and description*

Original model of Roman, Darby-Dowman, and Mitra:

$$\max_{\boldsymbol{x}\in X} \Gamma(\boldsymbol{x}), \qquad \text{where} \quad \Gamma(\boldsymbol{x}) := \min_{1\le i\le S} \Big(\operatorname{tail}_i(R\boldsymbol{x}) - \widehat{\tau}_i \Big).$$

Observation: for optimal portfolio $\boldsymbol{x}^*$, we usually have

$$\operatorname{tail}_1(R\boldsymbol{x}^*) - \widehat{\tau}_1 < \operatorname{tail}_i(R\boldsymbol{x}^*) - \widehat{\tau}_i \qquad (i = 2, \ldots, S).$$

Scaling of tails needed:

$$\max_{\boldsymbol{x}\in X} \widetilde{\Gamma}(\boldsymbol{x}), \qquad \text{where} \quad \widetilde{\Gamma}(\boldsymbol{x}) := \min_{1\le i\le S} \frac{1}{i}\Big(\operatorname{tail}_i(R\boldsymbol{x}) - \widehat{\tau}_i \Big).$$

Enhanced multi-objective SSD model: *Motivation and description*

$$\max_{\boldsymbol{x}\in X} \widetilde{\Gamma}(\boldsymbol{x}), \qquad \text{where} \quad \widetilde{\Gamma}(\boldsymbol{x}) := \min_{1\le i\le S} \frac{1}{i}\left(\text{tail}_i\left(R_{\boldsymbol{x}}\right) - \widehat{\tau}_i \right).$$

Scaled-tails model can be formulated as

$$\max \vartheta$$

$$\text{such that} \quad \vartheta \in \mathbb{R}, \quad \boldsymbol{x} \in X,$$

$$R_{\boldsymbol{x}} \succeq_{SSD} \widehat{R} + \vartheta.$$

Proof: $\quad \text{tail}_i\left(\widehat{R} + \vartheta\right) = \underbrace{\text{tail}_i\left(\widehat{R}\right)}_{\widehat{\tau}_i} + i\,\vartheta.$

Enhanced multi-objective SSD model: *Computational study*

Test data: 76 stocks from FTSE 100,
weekly observations during the period January 1993 - December 2003.
(A set of 565 pieces of data for each stock, and also for the FTSE 100 stock index.)

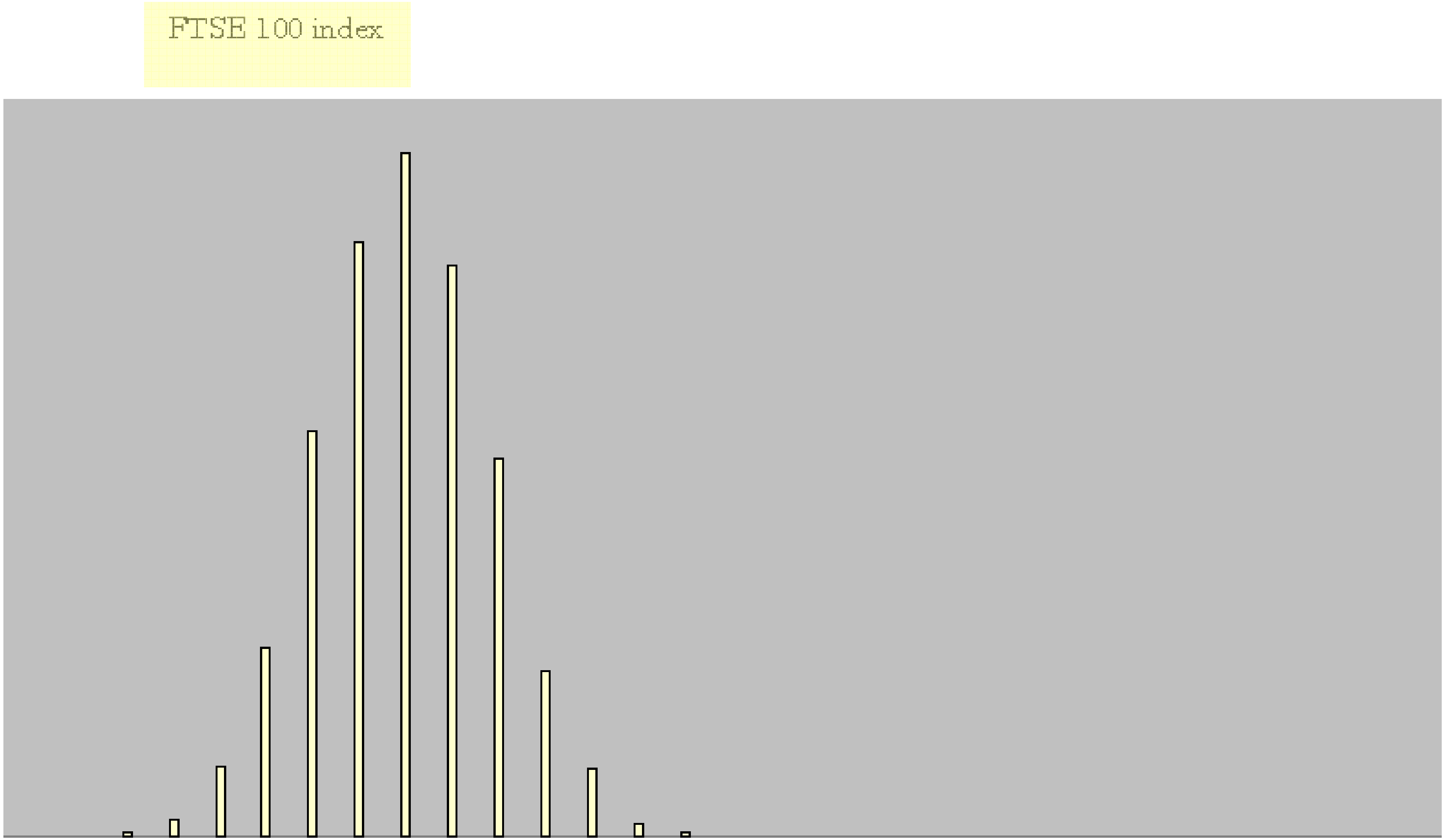
Scenarios generated by geometric Brownian motion,
parameters fitted to historical weekly returns.

Scenario sets of cardinality $5,000 \quad 7,000 \quad 10,000 \quad 15,000 \quad 20,000 \quad 30,000$.

Benchmark distribution: distribution of the FTSE 100 index. Not SSD-efficient.
Optimal portfolio return distribution dominates benchmark distribution.

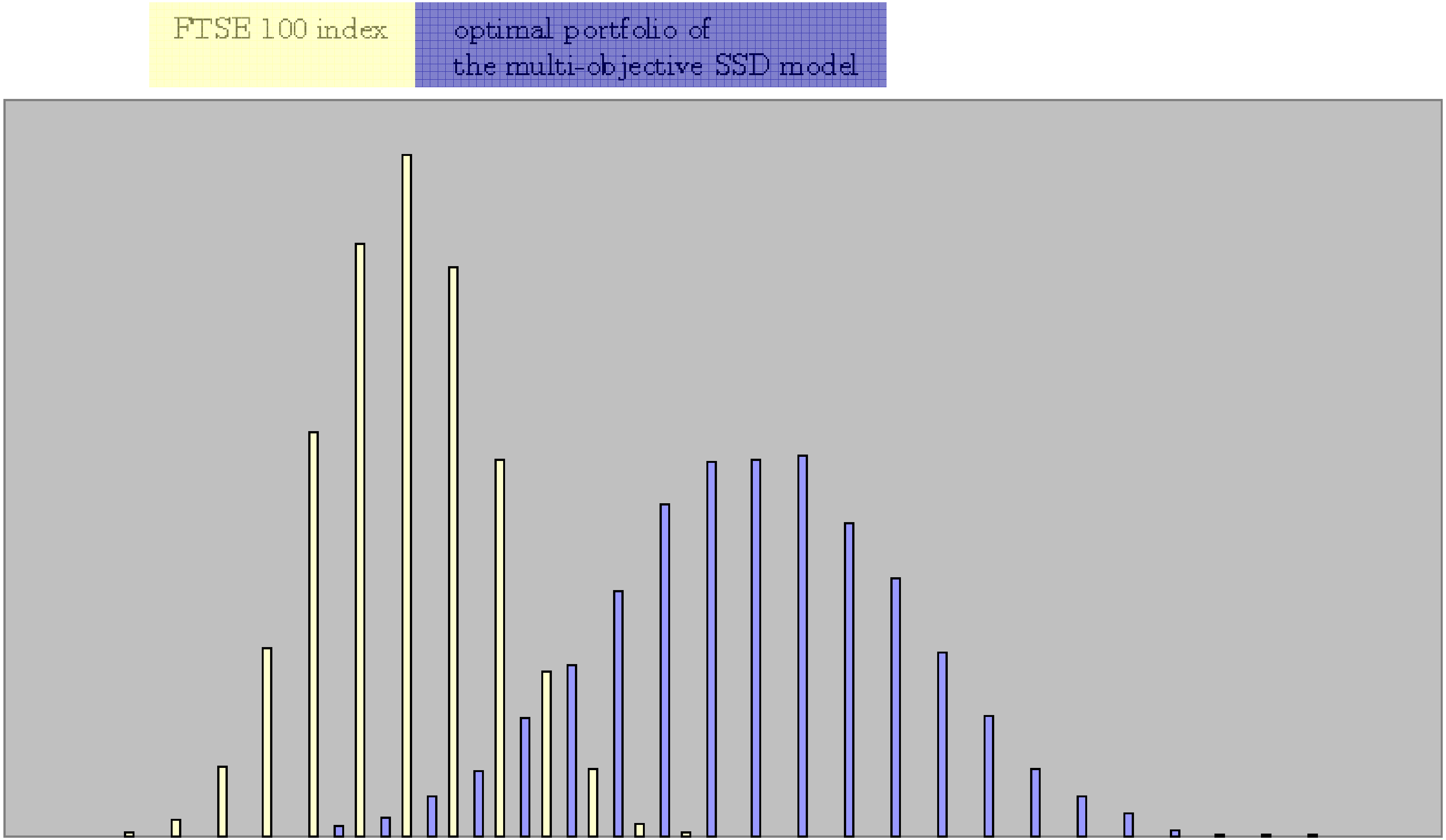
Enhanced multi-objective SSD model: *Computational study*

Return distributions represented by histograms (30,000 scenarios)



Enhanced multi-objective SSD model: *Computational study*

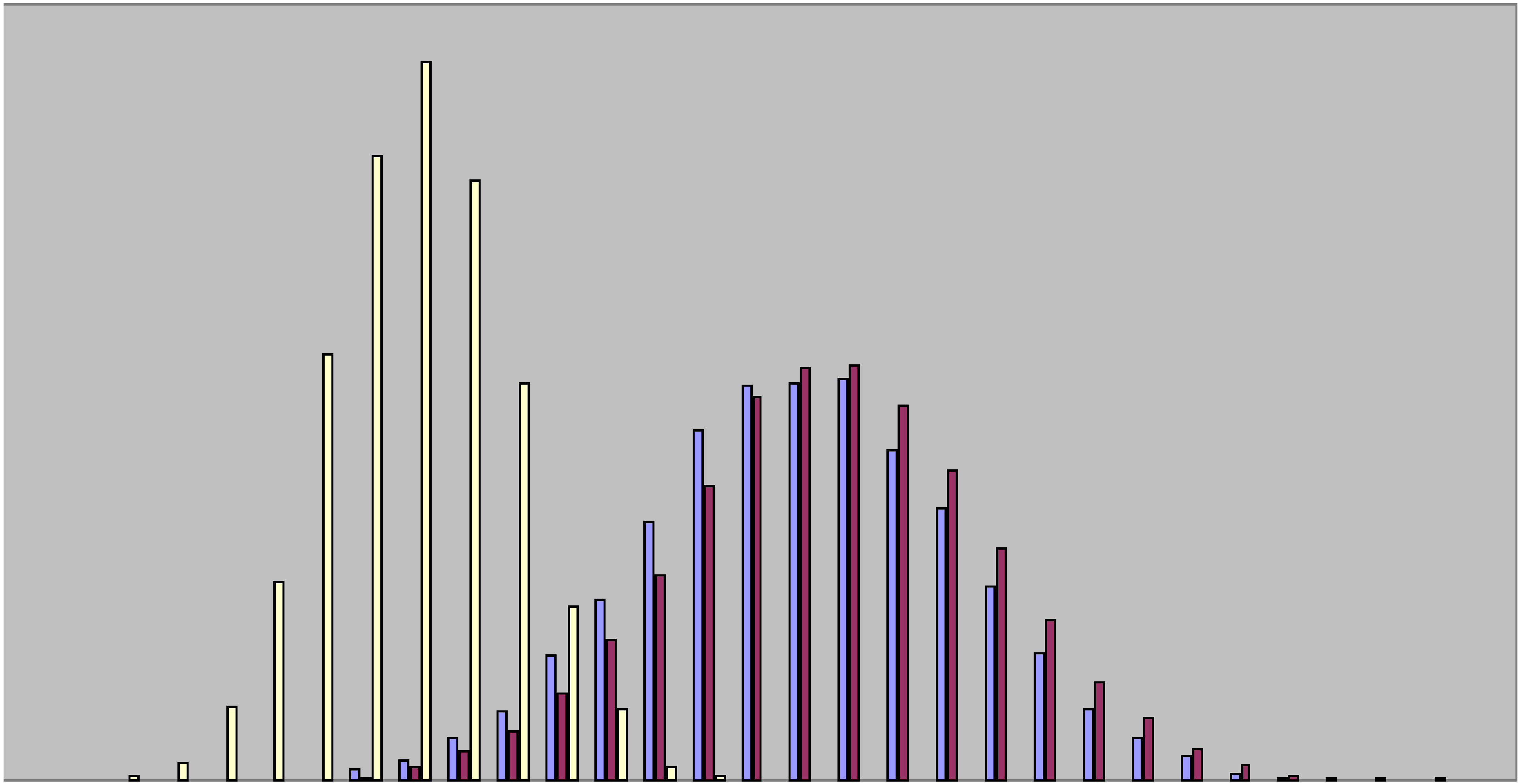
Return distributions represented by histograms (30,000 scenarios)



Enhanced multi-objective SSD model: *Computational study*

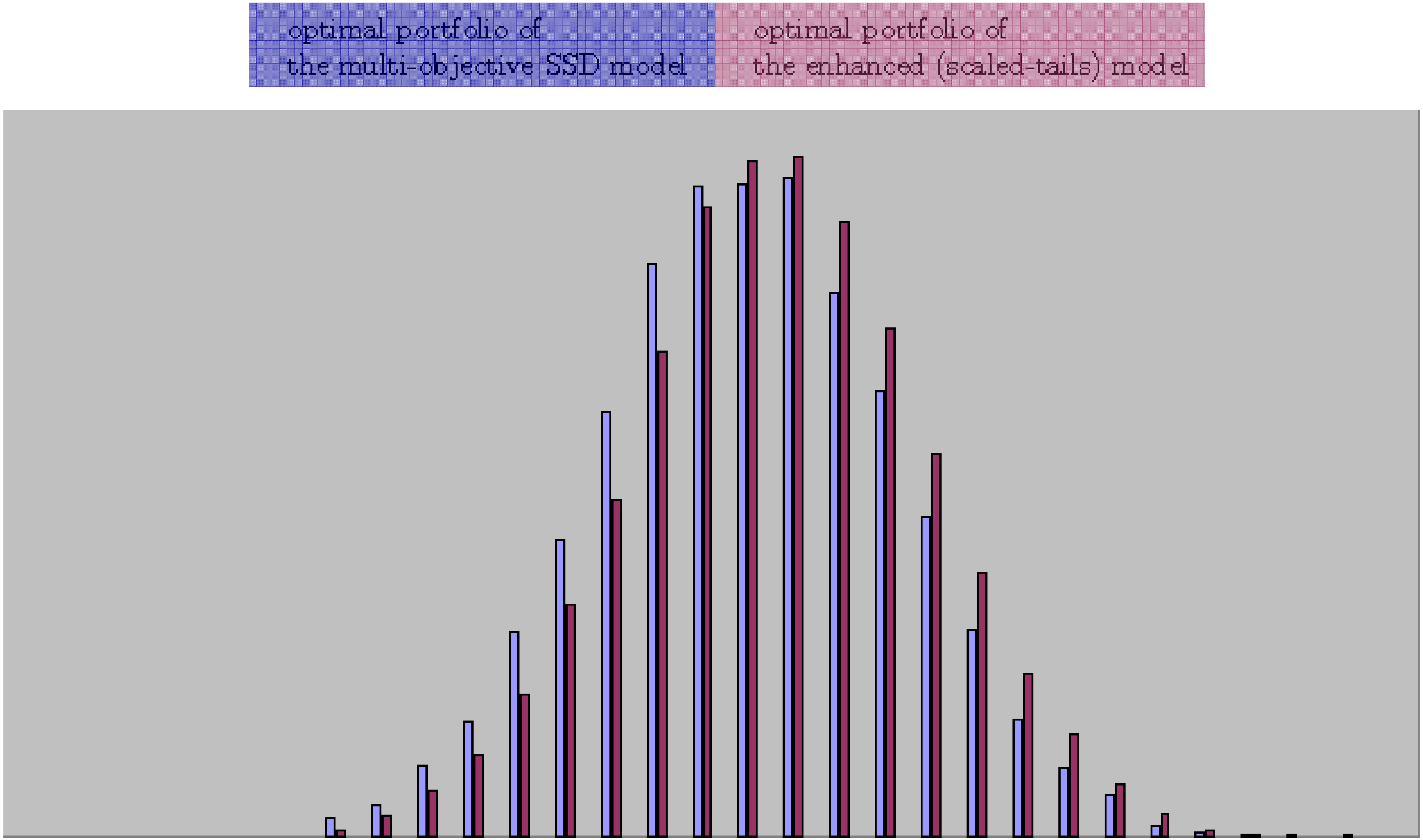
Return distributions represented by histograms (30,000 scenarios)

FTSE 100 index	optimal portfolio of the multi-objective SSD model	optimal portfolio of the enhanced (scaled-tails) model



Enhanced multi-objective SSD model: *Computational study*

Return distributions represented by histograms (30,000 scenarios)



Enhanced multi-objective SSD model: *Connection to SSD-constraints*

Scaled-tails model:

$$\max \vartheta$$

$$\text{such that} \quad \vartheta \in \mathbb{R}, \quad \boldsymbol{x} \in X,$$

$$R\boldsymbol{x} \succeq_{SSD} \widehat{R} + \vartheta.$$

SSD-constrained model of Dentcheva and Ruszczyński:

$$\max \mathrm{E}\left(R\boldsymbol{x}\right)$$

$$\text{such that} \quad \boldsymbol{x} \in X,$$

$$R\boldsymbol{x} \succeq_{SSD} \widehat{R}.$$

A solution method for the scaled-tails problem can be used
to solve the Lagrangian of the SSD-constrained problem.

Enhanced multi-objective SSD model: *Formulation with convex risk measure*

Scaled-tails model:

$$\max \vartheta$$

$$\text{such that} \quad \vartheta \in \mathbb{R}, \quad \boldsymbol{x} \in X,$$

$$R\boldsymbol{x} \succeq_{SSD} \widehat{R} + \vartheta.$$

Enhanced multi-objective SSD model: *Formulation with convex risk measure*

Scaled-tails model:

$$\max \vartheta$$

$$\text{such that} \quad \vartheta \in \mathbb{R}, \quad \boldsymbol{x} \in X,$$

$$R\boldsymbol{x} \succeq_{sso} \widehat{R} + \vartheta.$$

Equivalent formulation:

$$\min \varrho$$

$$\text{such that} \quad \varrho \in \mathbb{R}, \quad \boldsymbol{x} \in X,$$

$$R\boldsymbol{x} + \varrho \succeq_{sso} \widehat{R}.$$

Enhanced multi-objective SSD model: *Formulation with convex risk measure*

Scaled-tails model:

$$\max \vartheta$$

$$\text{such that} \quad \vartheta \in \mathbb{R}, \quad \boldsymbol{x} \in X,$$

$$R\boldsymbol{x} \succeq_{sso} \widehat{R} + \vartheta.$$

Equivalent formulation:

$$\min \varrho$$

$$\text{such that} \quad \varrho \in \mathbb{R}, \quad \boldsymbol{x} \in X,$$

$$R\boldsymbol{x} + \varrho \succeq_{sso} \widehat{R}.$$

Compact formulation:

$$\min_{\boldsymbol{x} \in X} \widehat{\rho}(R\boldsymbol{x}), \quad \text{where} \quad \widehat{\rho}(R) := \min \left\{ \varrho \in \mathbb{R} \;\middle|\; R + \varrho \succeq_{sso} \widehat{R} \right\}.$$

Theory of risk measures

Artzner, Delbaen, Eber, Heath (1999) ; Delbaen (2002).

Convex risk measures

Heath (2000) ; Carr, Geman, Madan (2001) ; Föllmer, Schied (2002).

Rockafellar, Uryasev, Zabarankin (2002-2006) ; Rockafellar (2007).

A risk measure ρ is convex if it satisfies the following criteria:

Convexity: $\rho\big(\lambda R + (1-\lambda)R'\big) \leq \lambda\rho(R) + (1-\lambda)\rho(R')$ for random returns R, R', and $0 \leq \lambda \leq 1$.

Monotonicity: $\rho(R) \leq \rho(R')$ for random returns R, R', $R \geq R'$.

Translation equivariance: $\rho(R + \varrho) = \rho(R) - \varrho$ for random return R, and $\varrho \in \mathbb{R}$.

Enhanced multi-objective SSD model: *Formulation with convex risk measure*

$$\widehat{\rho}(R) := \min\left\{ \varrho \in \mathbb{R} \;\middle|\; R + \varrho \succeq_{ssd} \widehat{R} \right\}$$ is a convex risk measure.

Using dual representation of convex risk measure: worst-case analysis.

Connection with robust optimisation.

Conclusion

Cutting-plane approach for the multi-objective model of Roman, Darby-Dowman, Mitra

3 orders of magnitude faster than direct approach (for $S = 600$).

No limit for the number of scenarios.

Effect of regularisation

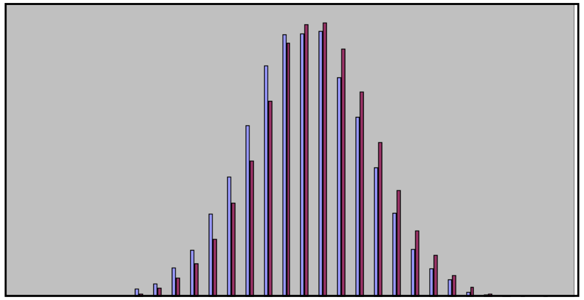
Iteration count reduces to one-third (for $n = 76$, $\epsilon = 1e - 7$, $S \leq 30,000$).

Very good scale-up properties.

Iteration count: $O\left(n \ln \frac{1}{\epsilon} \right)$.

Enhanced model

Optimal portfolio has superior return distribution.



Connection with robust optimisation.

Future prospects

Dual method for SSD-constrained problems.

Variance taken into account.

Two-stage generalization.

Prospective work: *Dual method for SSD-constrained problems*

Generalised version of the SSD-constrained model of Dentcheva and Ruszczyński:

$$\max \; \mathrm{E}(R\boldsymbol{x})$$

$$\text{such that} \quad \boldsymbol{x} \in X,$$

$$R\boldsymbol{x} \succeq_{SSD} \widehat{R} + \gamma,$$

where $\gamma \in \mathbb{R}$ given parameter.

Prospective work: ***Dual method for SSD-constrained problems***

Generalised version of the SSD-constrained model of Dentcheva and Ruszczyński:

$\max \, \mathrm{E}(R\boldsymbol{x})$	$\max \, \mathrm{E}(R\boldsymbol{x})$
such that $\quad \boldsymbol{x} \in X,$	such that $\quad \boldsymbol{x} \in X,$
$R\boldsymbol{x} \succeq_{SSD} \widehat{R} + \gamma,$	$\widehat{\rho}(R\boldsymbol{x}) \leq \gamma,$

where $\gamma \in \mathbb{R}$ given parameter.

Prospective work: *Dual method for SSD-constrained problems*

Generalised version of the SSD-constrained model of Dentcheva and Ruszczyński:

$\max \ \mathrm{E}(R\boldsymbol{x})$	$\max \ \mathrm{E}(R\boldsymbol{x})$
such that $\quad \boldsymbol{x} \in X,$	such that $\quad \boldsymbol{x} \in X,$
$R\boldsymbol{x} \succeq_{SSD} \widehat{R} + \gamma,$	$\widehat{\rho}(R\boldsymbol{x}) \leq \gamma,$

where $\gamma \in \mathbb{R}$ given parameter.

In practice, the parameter γ is set by a decision maker.

We can help them by approximating the efficient frontier.

Solution of Lagrangian problems

$$\max_{\boldsymbol{x} \in X} \mathrm{E}(R\boldsymbol{x}) - \lambda \widehat{\rho}(R\boldsymbol{x})$$

with different values of $\lambda \geq 0$.

Prospective work: ***Variance taken into account***

Mean-risk models using two risk measures, Roman, Mitra, Darby-Dowman (2006):

Portfolio represented by *(expectation, variance, CVaR)* of portfolio yield.

Approximation of efficient frontier constructed.

Out-of-sample analysis: superior performance of non-extremal efficient portfolios.

Proposed extension of multi-objective model:

Represent portfolio by *(expectation, variance,* $\widehat{\rho}$*)* of portfolio yield.

Approximation of efficient frontier to be constructed by cutting-plane approach.

Prospective work: *Two-stage SSD models*

2 time periods,

we can rebalance our portfolio at the beginning of each period,

let us compare *benchmark yield* and *portfolio yield* at the end of the second period.